Exercise 13

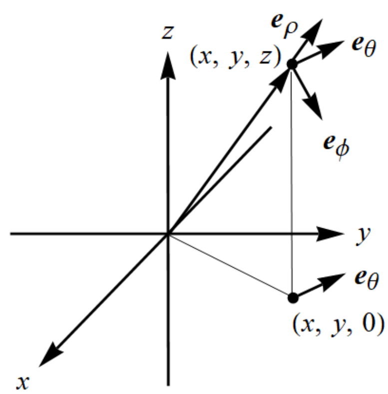
Using spherical coordinates and the orthonormal (orthogonal normalized) vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ [see Figure 1.4.8(b)],

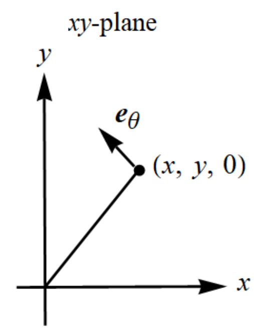
(a) express each of $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and (x, y, z); and

(b) calculate $\mathbf{e}_\theta \times \mathbf{j}$ and $\mathbf{e}_\phi \times \mathbf{j}$ both analytically and geometrically.

Solution

The relevant part of Figure 1.4.8 is shown here.





Start by calculating the radial unit vector.

$$\mathbf{e}_\rho = \frac{\boldsymbol{\rho}}{\|\boldsymbol{\rho}\|} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} = \frac{(x, y, z)}{\rho} = \left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right) = \frac{x}{\rho}\mathbf{i} + \frac{y}{\rho}\mathbf{j} + \frac{z}{\rho}\mathbf{k}$$

The azimuthal unit vector is perpendicular to the radial unit vector in the xy-plane, which means their dot product is zero.

$$\mathbf{e}_\theta \cdot \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} = 0 \quad \Rightarrow \quad \mathbf{e}_\theta = \frac{(\pm y, \mp x, 0)}{\sqrt{x^2 + y^2}}$$

Since $\mathbf{e}_\theta$ points to the upper left, we choose

$$\mathbf{e}_\theta = \frac{(-y, x, 0)}{\sqrt{x^2 + y^2}} = \frac{(-y, x, 0)}{r} = \left(-\frac{y}{r}, \frac{x}{r}, 0\right) = -\frac{y}{r}\mathbf{i} + \frac{x}{r}\mathbf{j}.$$

Finally, the polar unit vector can be obtained by taking the cross product of $\mathbf{e}_\theta$ and $\mathbf{e}_\rho$.

$$\mathbf{e}_\phi = \mathbf{e}_\theta \times \mathbf{e}_\rho = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{y}{r} & \frac{x}{r} & 0 \\ \frac{x}{\rho} & \frac{y}{\rho} & \frac{z}{\rho} \end{vmatrix} = \frac{1}{r\rho} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -y & x & 0 \\ x & y & z \end{vmatrix} = \frac{1}{r\rho}[xz\mathbf{i} + yz\mathbf{j} - (x^2 + y^2)\mathbf{k}] = \frac{xz}{r\rho}\mathbf{i} + \frac{yz}{r\rho}\mathbf{j} - \frac{r}{\rho}\mathbf{k}$$

Use these formulas to determine the desired cross products.

$$\begin{aligned}
\mathbf{e}_\theta \times \mathbf{j} &= \left(-\frac{y}{r}\mathbf{i} + \frac{x}{r}\mathbf{j}\right) \times \mathbf{j} \\
&= -\frac{y}{r}(\mathbf{i} \times \mathbf{j}) + \frac{x}{r}(\mathbf{j} \times \mathbf{j}) \\
&= -\frac{y}{r}(\mathbf{k}) \\
&= -\frac{y}{r}\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_\phi \times \mathbf{j} &= \left(\frac{xz}{r\rho}\mathbf{i} + \frac{yz}{r\rho}\mathbf{j} - \frac{r}{\rho}\mathbf{k}\right) \times \mathbf{j} \\
&= \frac{xz}{r\rho}(\mathbf{i} \times \mathbf{j}) + \frac{yz}{r\rho}(\mathbf{j} \times \mathbf{j}) - \frac{r}{\rho}(\mathbf{k} \times \mathbf{j}) \\
&= \frac{xz}{r\rho}(\mathbf{k}) - \frac{r}{\rho}(-\mathbf{i}) \\
&= \frac{r}{\rho}\mathbf{i} + \frac{xz}{r\rho}\mathbf{k}
\end{aligned}$$